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The phase configuration of internal waves around a body moving in a density stratified fluid

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A body is started impulsively from rest and moves in a curved path in a stably stratified fluid. The phase configuration of the internal waves which are generated is studied using small amplitude wave theory. The theory is compared with experiment for a few special cases which include a horizontal cylinder moving in a circular path in a vertical plane, oscillating through a large amplitude in a horizontal plane, and moving with constant velocity past stationary plates. Theory and experiment show reasonable agreement except where the waves produced by the wake dominate the flow.

1. Introduction

The steady internal waves which develop around a body moving with a constant velocity in a fluid with a constant Brunt–Väisälä frequency were discussed by Lighthill (1967), Mowbray & Rarity (1967*b*) and Rarity (1967). If an oscillating body moves at a constant mean velocity then an oscillatory wave system is superimposed on the steady wave system (Stevenson 1969; Stevenson & Thomas 1969). The theoretical and experimental work is now extended to the waves around a body moving in a curved path. Small amplitude wave theory is used to obtain an equation for the phase configuration of the waves in terms of a body velocity which can vary with time. This equation reduces to that of Stevenson & Thomas (1969) when the time and the radius of curvature approach infinity and when the body velocity is constant. The radiation condition is included in the analysis and there is no need to study the wavenumber surfaces explicitly. The waves do not maintain the geometrical similarity which existed in the previous work.

The phase configurations of the waves around a horizontal cylinder are derived for a few specific cases: (*a*) an oscillating cylinder which moves in a circular path with a constant mean angular velocity, (*b*) a cylinder which oscillates through a large amplitude in a horizontal plane and (*c*) a cylinder which moves with constant velocity past a stationary plate. The theory is compared with experiments in stratified brine and the wave system is viewed using a schlieren system. It is found that Cauchy–Poisson waves due to the impulsive start are present in many of the wave systems. Consequently the theory for these waves is also outlined and compared with experiment.

2. Analysis

The dispersion relation for incompressible inviscid internal gravity waves generated by an oscillating point disturbance is (Lighthill 1967)

$$\omega^2(l^2+m^2+n^2) = \omega_0^2(l^2+m^2). \tag{1}$$

ω_0 is the Brunt–Väisälä frequency, which will be assumed constant, and ω is the frequency associated with the energy propagating from the point disturbance. The wavenumber $\mathbf{k}$ has components (l, m, n) in the (x, y, z) directions with x and y in the horizontal plane and z in the upward vertical direction. We are interested in short-wave solutions with $l^2+m^2+n^2 \gg \omega_0^2/2g$ and have omitted a term in $\omega_0^2/2g$ from the dispersion relation. (g is the accleration due to gravity.) Further qualification is required when considering the Cauchy–Poisson waves but this will be discussed later.

At sufficiently large distances from the forcing region the principle of stationary phase applied to the integral equation for the properties within the waves shows that energy propagates away from the disturbance with the group velocity $\mathbf{c} = \nabla_{\mathbf{k}}\omega$. For internal waves, from (1),

$$\mathbf{c} = (\omega_0^2 n/\omega k^4)\,(ln, mn, -(l^2+m^2)), \tag{2}$$

where $k = |\mathbf{k}|$. The phase configuration of the waves around an oscillating body moving with a constant mean velocity was evaluated by Stevenson & Thomas (1969) using this group velocity. The waves agreed quite well over the whole flow field except, perhaps, very close to the body. Therefore to simplify the present problem the group-velocity concept will be applied to the whole flow field.

At time $t = t_0$ a body starts to move with a velocity $\mathbf{Q}(t)$ such that its distance from an origin O fixed in the undisturbed background fluid is $\mathbf{R}(t)$ as in figure 1. If the body is at the point A at time $t = t_1$ then energy will propagate from A with a velocity $\mathbf{c}$ and at a later time t will be at a position P which is at a distance

$$\mathbf{r} = \mathbf{R}_1 + (t-t_1)\,\mathbf{c} \tag{3}$$

from the origin. The subscript 1 is used to denote conditions at time t_1.

If ω_f is the frequency associated with either the path of the body, vortex shedding from the body or with forced oscillations of the body, then the relation between ω and ω_f is given by the Doppler equation

$$\omega = \omega_f + \mathbf{Q}_1 . \mathbf{k}. \tag{4}$$

Any number of forced frequencies may be included merely by superimposing the wave system due to each individual value of ω_f.

The phase Φ at P is given by

$$\Phi = (\mathbf{k}.\mathbf{c}-\omega)\,(t-t_1)-\omega_f t_1+\phi_0, \tag{5}$$

where ϕ_0 is a constant. From (2) it may be shown that $\mathbf{k}.\mathbf{c} = 0$, so that (5) may be rearranged to read

$$t-t_1 = (\phi_0-\Phi-\omega_f t)/(\omega-\omega_f). \tag{6}$$

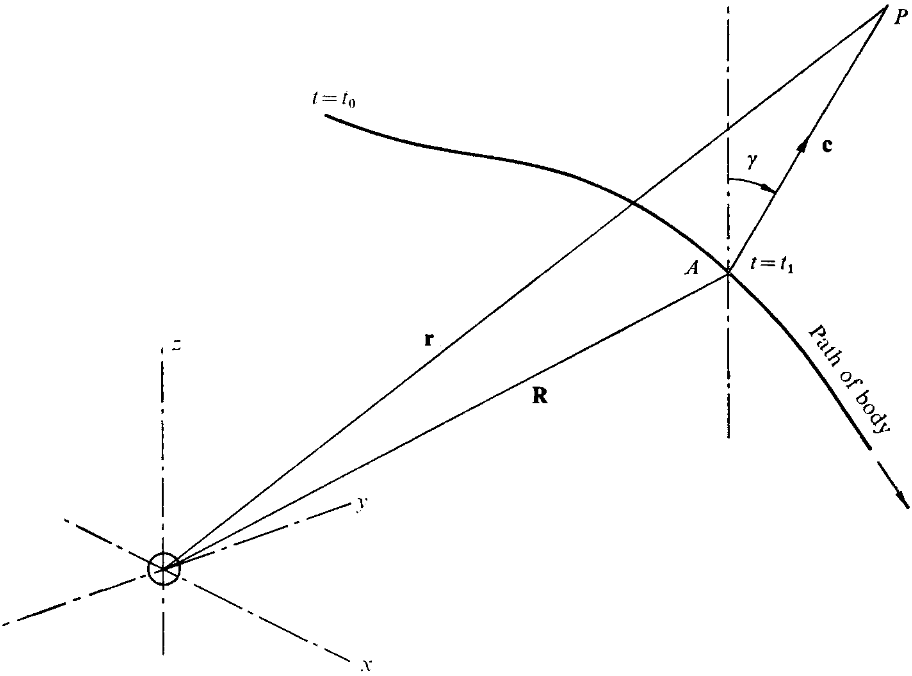


FIGURE 1. The path of the body.

The radiation condition implies that $t-t_1 > 0$ and therefore

$$t-t_1 = |N/(\omega_0|\cos\gamma| \pm \omega_f)|, \tag{7}$$

where $\cos^2\gamma = (\omega/\omega_0)^2$ and γ is the angle between the vertical and the group-velocity vector as in figure 1. N varies by 2π between one wave crest and the next.

Equations (2) and (7) are substituted into (3) to give

$$\mathbf{r} = \mathbf{R}_1 + \left|\frac{N}{M}\right| \frac{\omega_0 n}{\omega k^4}(ln, mn, -(l^2+m^2)), \tag{8}$$

where $M = |\cos\gamma| \pm \omega_f/\omega_0$. The phase configuration can be evaluated using this equation together with the dispersion relation and (4).

The present experiments are restricted to two-dimensional waves and in this case the equation for the phase configuration is

$$\mathbf{r} = (x,z) = \mathbf{R}_1 + \left|\frac{N}{M}\right| \{U_1\cos\gamma - W_1\sin\gamma\}\frac{|\cos\gamma|}{\omega_0 M}(\sin\gamma\tan\gamma, \sin\gamma), \tag{9}$$

with $\mathbf{Q}_1 = (U_1, W_1)$.

Before this equation is compared with experiment there is one other wave system which must be considered because it is present in most of the experiments. This is the Cauchy–Poisson wave system which is generated by an impulsive disturbance. In the above equations the wavenumbers have been restricted to those which satisfy both the dispersion relation and the Doppler equation. However, in the Cauchy–Poisson waves the only restriction is that the wavenumbers generated must satisfy the dispersion relation. If the disturbance occurs at time $t = t_0$ the phase at a later time is given by

$$\Phi = (\mathbf{k}.\mathbf{c} - \omega)(t - t_0).$$

For internal waves $\mathbf{k}.\mathbf{c} = 0$ and so, making use of the dispersion relation in the form $\omega = \pm\omega_0 \cos\gamma$, we have $\Phi = \pm\omega_0(t-t_0)\cos\gamma$. Therefore the waves are St Andrew's crosses inclined at an angle γ to the vertical, where

$$\gamma = \cos^{-1}\{N/\omega_0(t-t_0)\}. \tag{10}$$

N varies by 2π between one wave crest and the next. Internal waves have a phase velocity directed towards the horizontal level of the disturbance. Thus the impulsive waves appear near the vertical and move towards the horizontal so that as time increases more and more waves are present.

Mowbray & Rarity (1967*a*) studied the Cauchy–Poisson problem and included the $\omega_0^2/2g$ term in the dispersion relation. Waves similar in shape to the Kelvin ship wave were obtained. The wavelength of the vertical transverse waves, which cross the horizontal plane of the disturbance, decreases as time increases. For the present experiments, 60 s after the impulse, the wavelength of these waves would have decreased to 22 m. This is a length scale very much greater than any in the experiments and the waves due to the $\omega_0^2/2g$ term were not observed. Consequently the experiments will be compared with (10), which is the same as that obtained when $\omega_0^2/2g \to 0$ in Mowbray & Rarity's theory.

3. Comparison of theory and experiment

3.1. *The apparatus*

A glass-sided tank 1·8 m long, 0·9 m high and 0·55 m from front to back was filled with stratified brine which had an almost constant Brunt–Väisälä frequency. A schlieren system with 0·46 m diameter mirrors was used to observe the waves which developed when a horizontal circular cylinder was moved through the fluid. The cylinder diameters were between 1·5 and 10 mm with ratios of length to diameter of approximately 40.

3.2. *Impulsive-start waves*

The waves produced when a circular cylinder 6 mm in diameter was impulsively moved through a distance of less than 5 mm are shown in figure 2 (plate 1). The schlieren photographs were taken through the sides of the tank and the black vertical line is the cylinder support. It is seen that the number of waves increases with time and that the waves are described very well by (10).

From (2) it can be shown that a locus of constant wavenumber consists of two circles which have the z axis as a tangent and whose centres are on the x axis at $\pm(t-t_0)\,\omega_0/2k$. In figure 2(*c*), no waves can be observed within regions roughly bounded by a constant-wavenumber locus. It appears either that the schlieren system cannot resolve wavenumbers higher than 0·4 mm^{-1}, or that the body did not generate these wavenumbers, or that the waves have been dissipated by viscous effects. Most of the energy seems to be in the range of wavenumbers between 0·4 and 0·04 mm^{-1}.

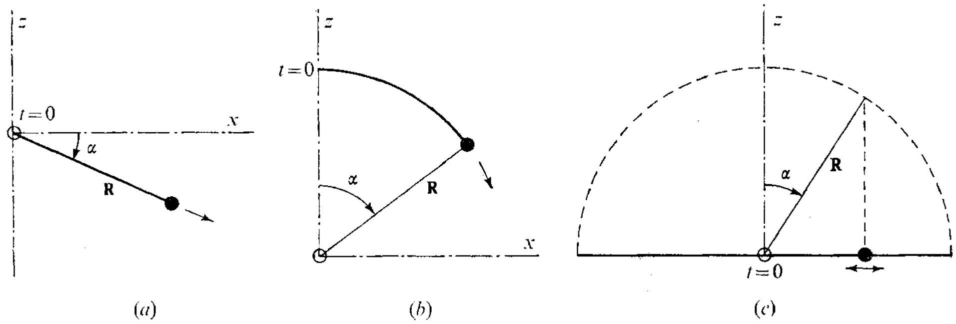


FIGURE 3. A horizontal cylinder moves (*a*) with a constant mean velocity at an angle α to the horizontal, (*b*) in a circular path and (*c*) with large amplitude oscillations in a horizontal plane.

3.3. *Cylinder moving with constant velocity*

If an oscillating body moves with a constant mean velocity at an angle α to the horizontal (figure 3*a*) then

$$\mathbf{Q}_1(x,z) = (Q\cos\alpha,\ -Q\sin\alpha) \quad \text{and} \quad \mathbf{R}_1(x,z) = (Qt_1\cos\alpha,\ -Qt_1\sin\alpha).$$

Therefore the phase configuration, equation (9), takes the form

$$\left(\frac{\omega_0 x}{Q}, \frac{\omega_0 z}{Q}\right) = \omega_0 t_1(\cos\alpha,\ -\sin\alpha) + \left|\frac{N}{M}\right| \cos(\gamma-\alpha)\frac{|\cos\gamma|}{M}(\sin\gamma\tan\gamma, \sin\gamma) \quad (11)$$

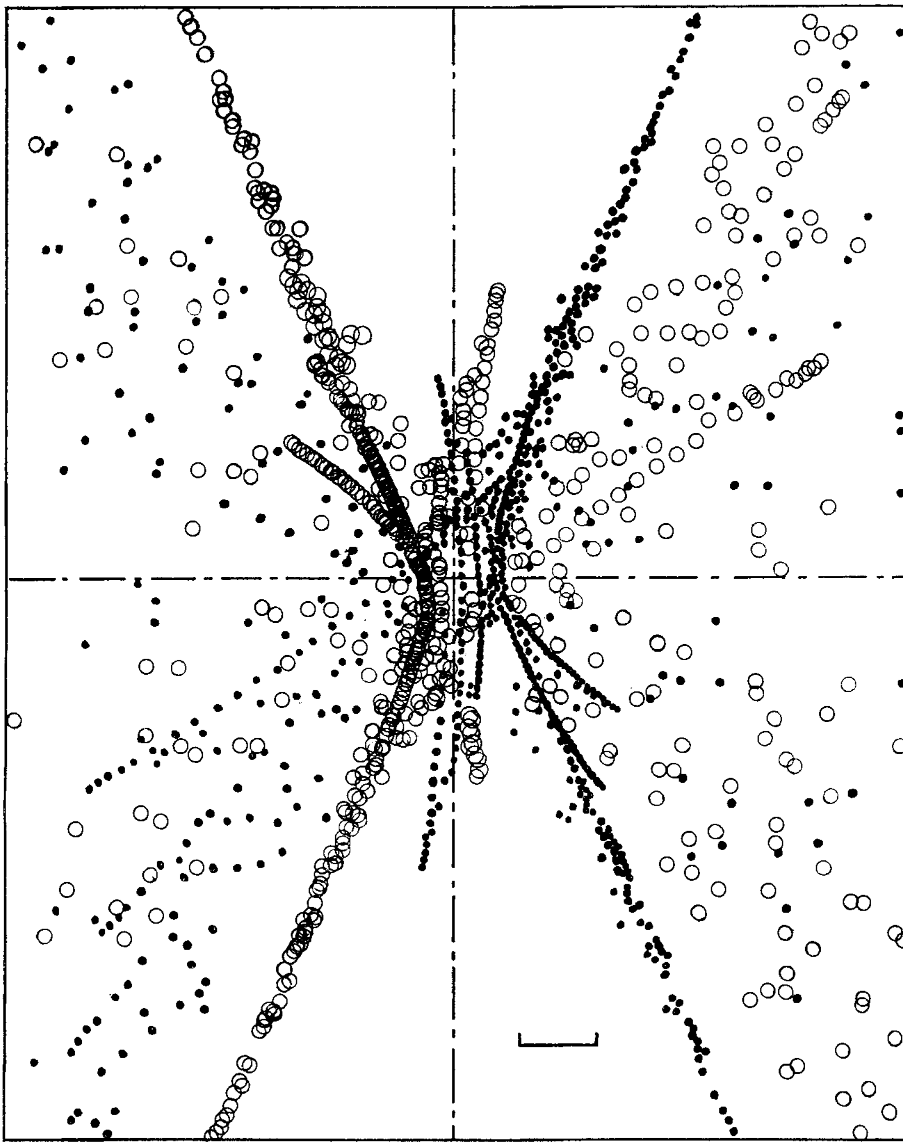
and $t_1 = (t - |N/M|/\omega_0)$, which is subject to the conditions $0 \leqslant t_1 \leqslant t$. If the body stops at time t_2, where $t_2 < t$, then $t_1 \leqslant t_2$. When $t \to \infty$ and after a suitable change of the co-ordinate system, equation (11) reduces to that given by Stevenson & Thomas (1969).

Figure 4 (*a*) (plate 2) shows the lee waves behind a cylinder moving horizontally from an impulsive start. The steady wave system obtained from (11) with $\omega_f = 0$ is a series of circular arcs centred on the body and contained within a circle passing through the body and through the starting position A. The impulsive-start waves, equation (10), are straight lines which pass through A and are tangential to the leading edges of the steady waves. Figure 4(*b*) (plate 2) shows how the wave system looks some time after the cylinder has stopped. In figure 4(*c*) (plate 2) it is shown how the impulsive-start waves are tangential to the leading edge of the steady waves generated by a body moving at 20° to the horizontal.

Figure 5 (plate 3) shows examples of the experimental and theoretical wave pattern which is obtained when a cylinder moves at constant velocity past an inclined plate. The reflected waves satisfy the following condition given by Mowbray & Rarity (1967*a*):

$$k_I \sin(\xi-\theta) = k_R \sin(\xi+\theta), \quad (12)$$

where the suffixes I and R refer to the incident and reflected waves respectively. The plate is inclined at an angle ξ to the horizontal and the incident and reflected waves make an angle θ with the horizontal.



(a)

FIGURE 8(a). For legend see facing page.

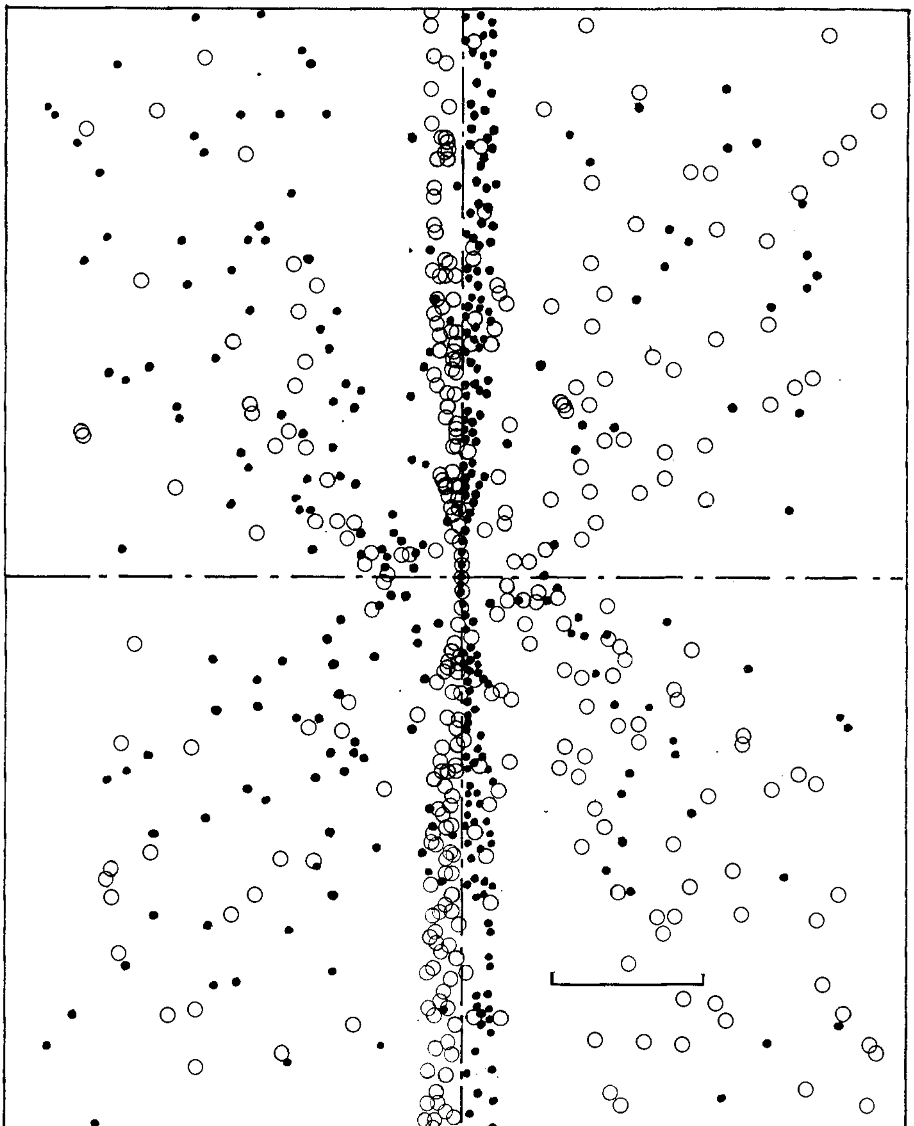
3.4. *Cylinder moving in circular path*

If a body moves in a circular path with a constant angular velocity ω_c (figure 3b) then $\mathbf{Q}_1 = (\omega_c R\cos\alpha, -\omega_c R\sin\alpha)$ and $\mathbf{R}_1 = (R\sin\alpha, R\cos\alpha)$. Lines of constant phase, from (9), are given by

$$\left(\frac{x}{R}, \frac{z}{R}\right) = (\sin\alpha, \cos\alpha) + \left|\frac{N}{M}\right|\frac{\omega_c}{\omega_0}\frac{|\cos\gamma|}{M}\cos(\gamma-\alpha)\,(\sin\gamma\tan\gamma, \sin\gamma), \tag{13}$$

where

$$\alpha = \omega_c t_1 = \left(t\omega_c - \left|\frac{N}{M}\right|\frac{\omega_c}{\omega_0}\right). \tag{14}$$



(*b*)

FIGURE 8. (*a*) A body moves in a circular path, equation (13), with $t\omega_c = 80$, $\omega_c/\omega_0 = 0{\cdot}92$ and $\omega_f = 0$. (*b*) A body oscillates in a horizontal plane, equation (16), with $t\omega_c = 50$, $\omega_c/\omega_0 = 1{\cdot}0$ and $\omega_f = 0$. The scale marks are of length $2R$. The circles and dots have a phase difference of π.

α is subject to the conditions

$$0 \leqslant t_0\omega_c < \alpha \leqslant t\omega_c. \tag{15}$$

If the body stops at time t_2, where $t_2 < t$, then $\alpha \leqslant t_2\omega_c$.

Schlieren photographs of the wave system around a horizontal cylinder moving in a circular path with constant angular velocity are reproduced in figure 6 (plate 4). The shape of the waves, computed by taking equal steps in γ, is shown as points in the figures or, when the points are close together, as a continuous line. The density of points does not represent energy density. For a true picture of the energy density it would be necessary to solve the equations satisfying the boundary conditions on the body.

A body moving vertically with constant velocity produces a herring-bone wave pattern and one travelling horizontally produces circular-arc waves. When the body moves in a circular path it is seen from figure 6 how these wave patterns gradually change from one to the other.

Theoretical and experimental waves around an oscillating body moving in a circular path are shown in figure 7 (plate 5). Wave patterns when the oscillatory frequency ω_f is above and below the Brunt–Väisälä frequency are shown.

When a body moves in a circular path with $\omega_f = 0$, the number of waves produced in one revolution depends only on the angular velocity. A change in the radius of the circle and a corresponding change in the body velocity merely produces a geometrical scaling of the whole wave system. After several revolutions the wave system is very complicated. However, it is interesting to see how the pseudo-steady waves (in which $\omega_f = 0$) develop into a cross when the angular velocity ω_c is close to the natural frequency ω_0. Equation (13), with $\omega_c/\omega_0 = 0{\cdot}92$ and $\omega_f = 0$, produces the picture shown in figure 8(*a*). Some photographs of the cross-waves produced when a horizontal cylinder moves in a clockwise circular path with $\omega_f = 0$ are presented in figures 9(*a*)–(*d*) (plate 6). The horizontal cylinder is producing quite a large wake and this spreads out to the right at the top of the path and to the left at the bottom. The wake and the waves are coupled and the wave system is much wider than that predicted by (13). The arms of the cross-wave have twice the width of those described by the viscous similarity analysis of Thomas & Stevenson (1972). In other words, there are twice as many visible displacement peaks in the waves shown in figure 9 as there are in the waves produced when a body oscillates with an amplitude which is less than the body diameter, the type of oscillation used by Mowbray & Rarity and Thomas & Stevenson.

3.5. *A cylinder oscillating with a large amplitude*

Finally we look at the waves produced when a body oscillates in a horizontal plane with an amplitude which is large compared with the diameter of the body. If the body is oscillating with frequency ω_c in a horizontal plane (figure 3*c*) then we can write $\mathbf{R}_1 = (R\sin\alpha, 0)$ and $\mathbf{Q}_1 = (\omega_c R\cos\alpha, 0)$, so that the phase configuration, equation (9), is

$$\left(\frac{x}{R}, \frac{z}{R}\right) = (\sin\alpha, 0) + \left|\frac{N}{M}\right| \frac{\omega_c}{\omega_0} \frac{|\cos\gamma|}{M} \cos\alpha \sin\gamma(\sin\gamma, \cos\gamma). \qquad (16)$$

α is again subject to the conditions (14) and (15).

When the frequency of oscillation ω_c is close to the natural frequency, (16) produces a cross similar to that for the circular path. When $\omega_c = \omega_0$ and $\omega_f = 0$ equation (16) produces the vertical wave shown in figure 8(*b*). Note that the inviscid, small amplitude, point-disturbance theory (Mowbray & Rarity 1967*a*) does not produce a wave when the frequency of oscillation is equal to ω_0, although the viscous similarity theory of Gordon & Stevenson (1972) does produce such a wave.

Photographs of a cylinder oscillating through a large amplitude are shown in figures 9(*e*) and (*f*) (plate 6). When ω_c is less than ω_0 two halves of a cross-wave are produced but each half is centred on the point where the body velocity is zero. The waves do not agree with (16) because the largest disturbance arises when the wake overtakes and engulfs the cylinder as the cylinder comes to rest at the end of each half oscillation.

In figure 9 the schlieren photographs show vortex shedding from the strut supporting the model. However, this does not affect the wave system significantly.

4. Conclusion

The phase configuration of the internal waves in a density stratified fluid has been described by two equations. One equation describes the energy, which, on leaving the disturbance region, satisfies a Doppler equation. The other describes the Cauchy–Poisson waves which are a result of an impulsive disturbance. The phase configuration calculated from these two equations has been compared with the wave system generated by a horizontal cylinder which moved along several specific paths. The theory and experiment agreed quite well providing the wake was not a strong source of waves. If the body moves through its own wake or if the wake moves over the body then extra sources must be included in the analysis.

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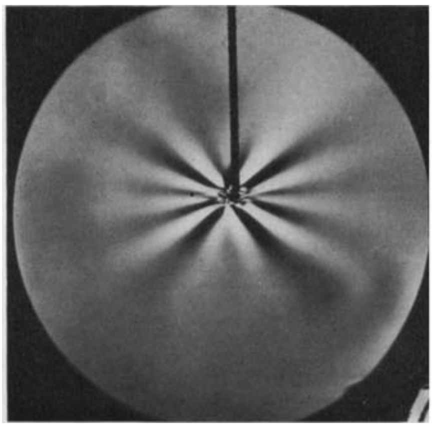

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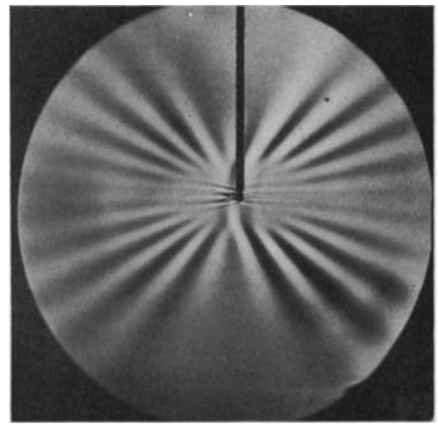

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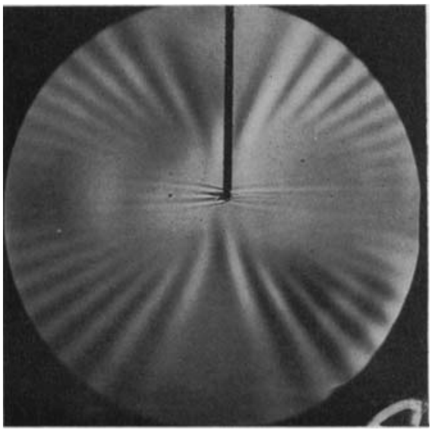


(c)

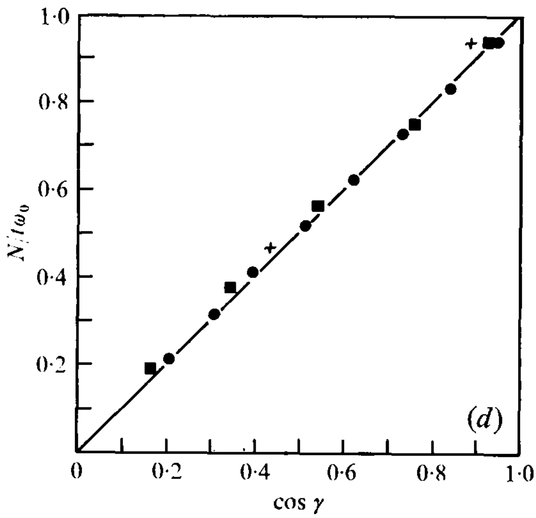
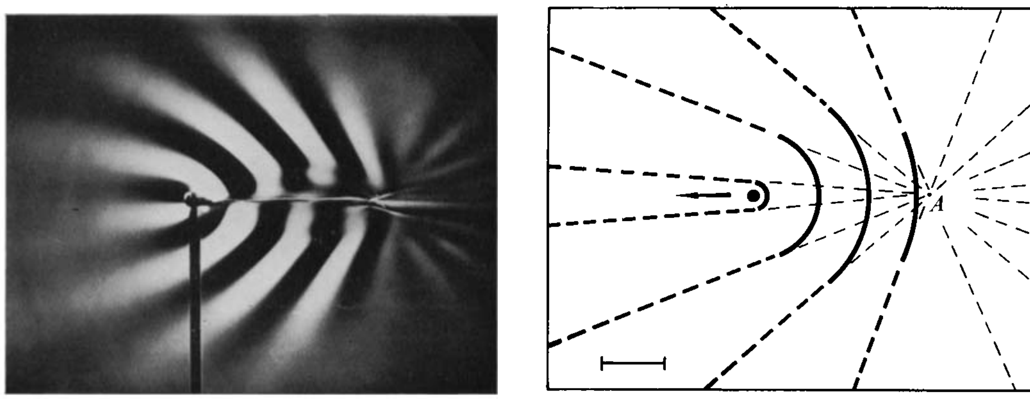


FIGURE 2. The cylinder is started and stopped at time $t = 0$. The schlieren photographs show the phase configuration of the impulsive waves at (a) $t = 10$ s, (b) $t = 25$ s and (c) $t = 45$ s. The angle γ which the wave crests make with the vertical is compared with (10) in (d). Experimental points: +, $t = 10$ s; ■, $t = 25$ s; ●, $t = 45$ s.



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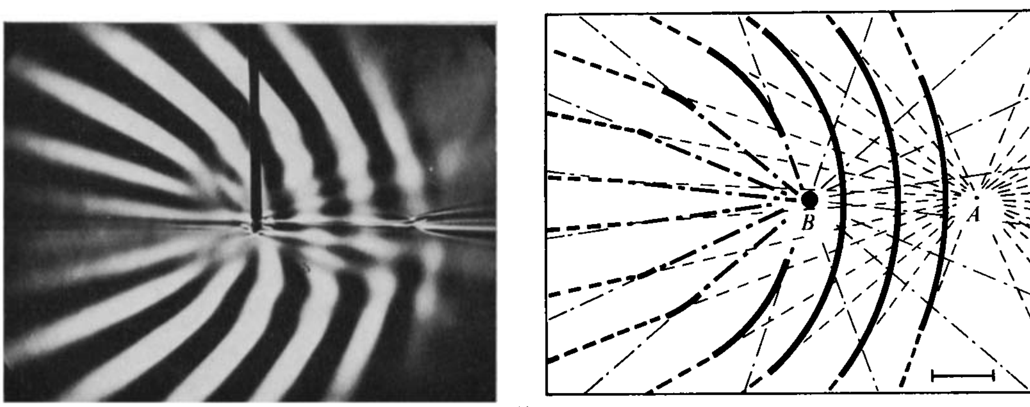

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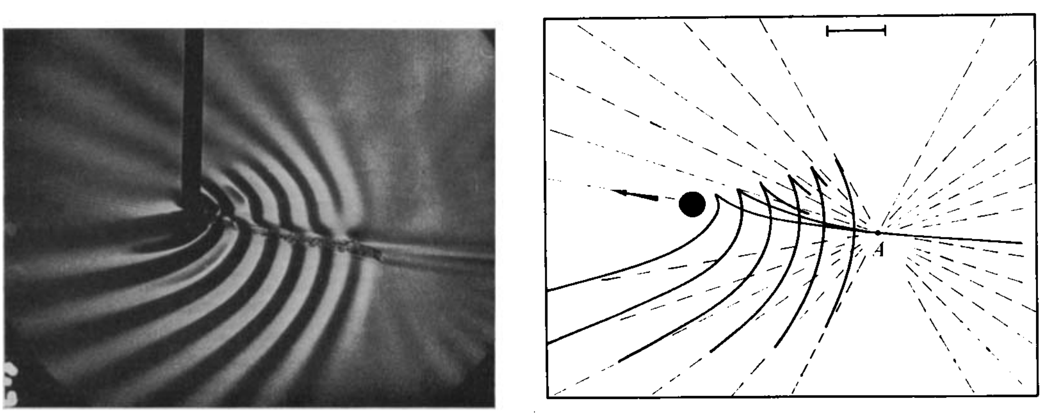

(*c*)

FIGURE 4. (*a*) The body moves horizontally with constant velocity from the point A. ——, steady lee waves, equation (11); – – –, impulsive-start waves, equation (10). $Q = 6{\cdot}6\ \mathrm{mm\ s^{-1}}$, $t_0 = 0$, $t = 22$ s. (*b*) The body moves horizontally with constant velocity starting from A and stopping at B. Wave pattern 22 s after the stop: ——, steady waves, equation (11); – – –, impulsive-start waves, equation (10); — - —, impulsive-stop waves, equation (10). $Q = 6{\cdot}6\ \mathrm{mm\ s^{-1}}$, $t_0 = 0$, $t_2 = 20$ s, $t = 42$ s. (*c*) The body moves from point A with constant velocity at an angle of 10° to the horizontal. ——, steady wave system; – – –, impulsive-start system. $Q = 4{\cdot}9\ \mathrm{mm\ s^{-1}}$. The scale lengths are 50 mm.

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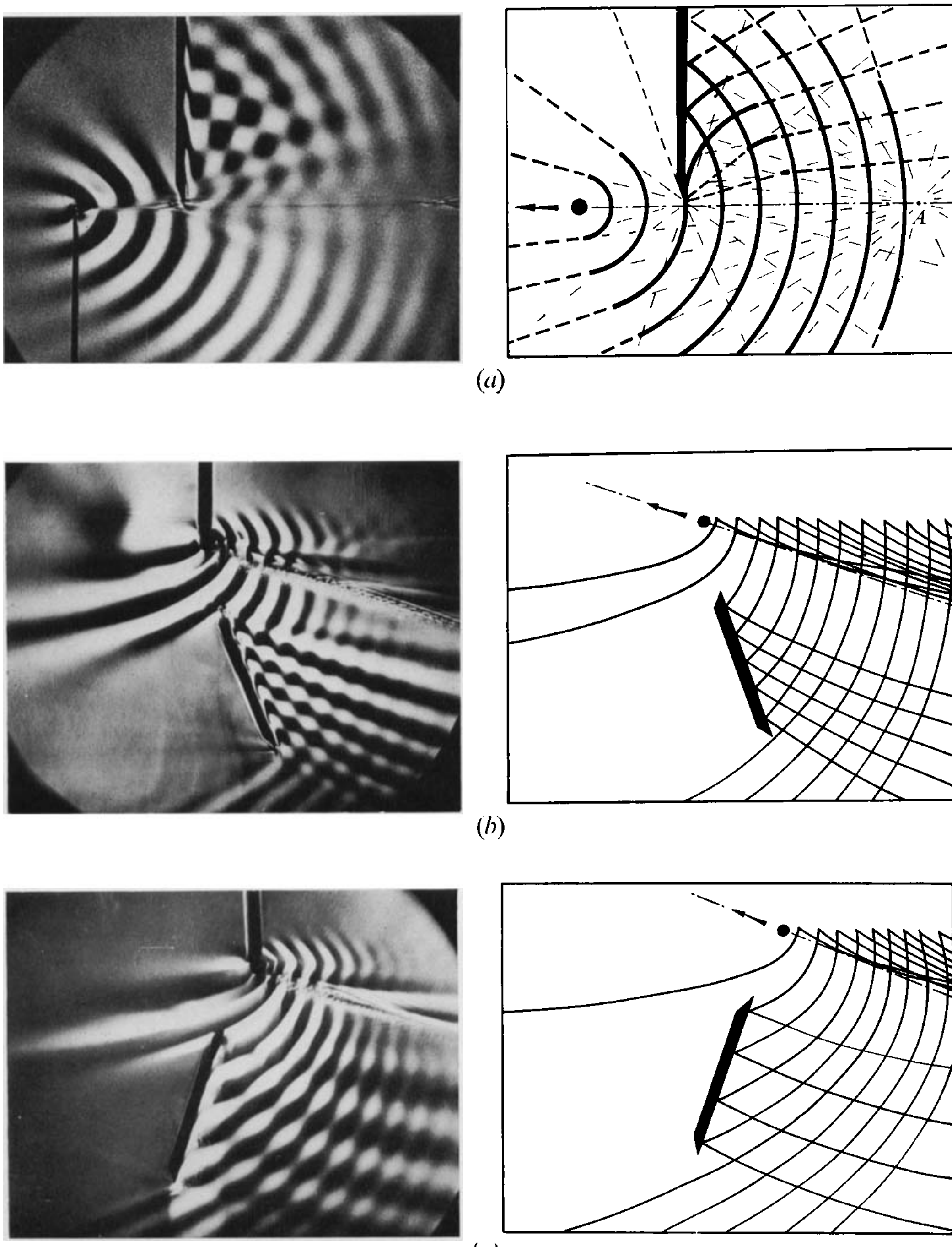
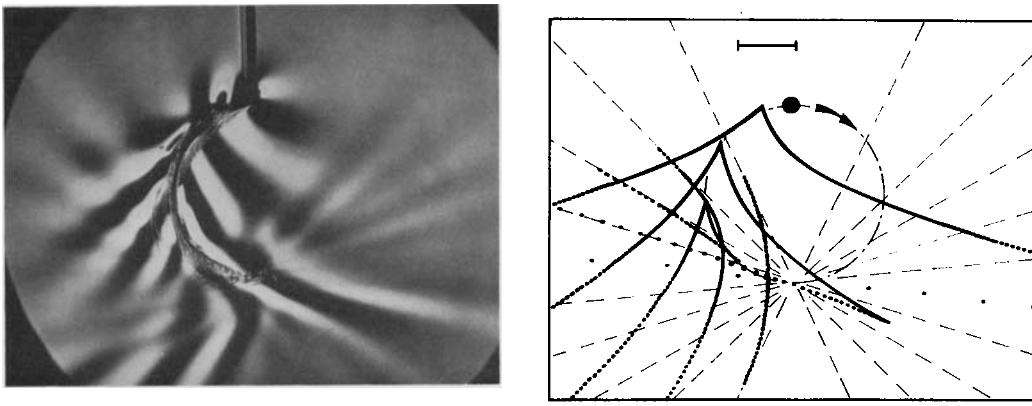


FIGURE 5. Reflected waves. The wave pattern produced when a horizontal cylinder moves at constant velocity past a stationary flat plate. (*a*) $\alpha = 0$, with a vertical plate. (*b*), (*c*) $\alpha = 20°$, with inclined plates.



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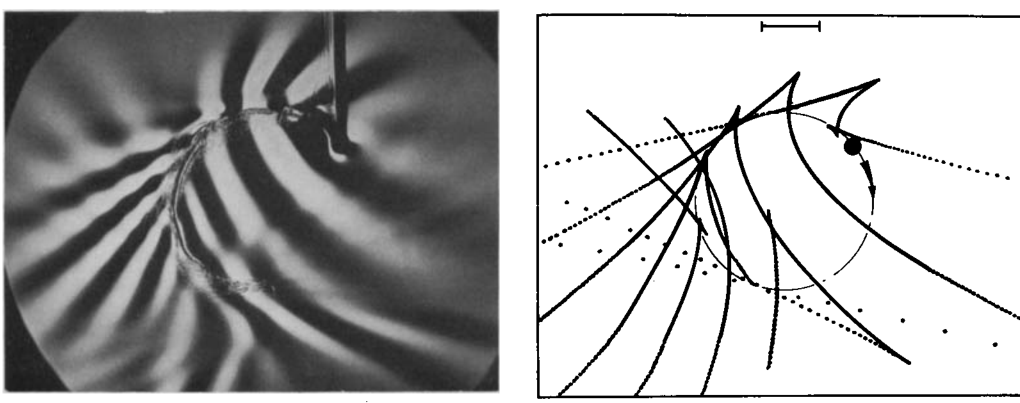

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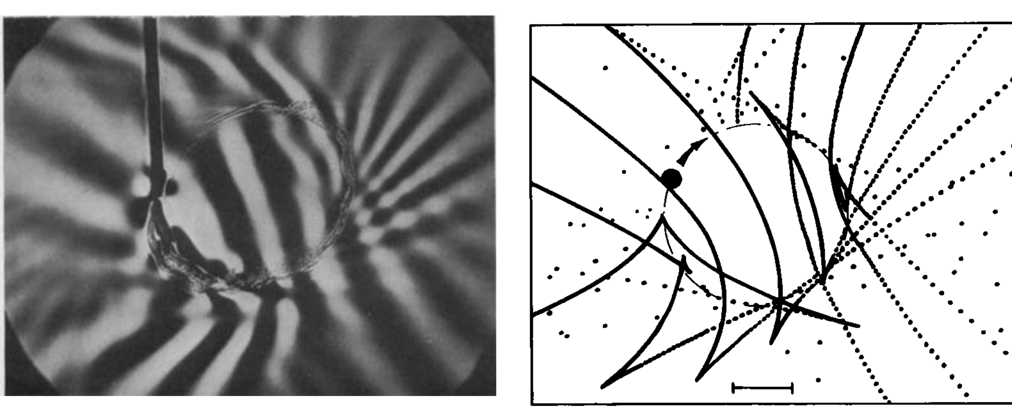
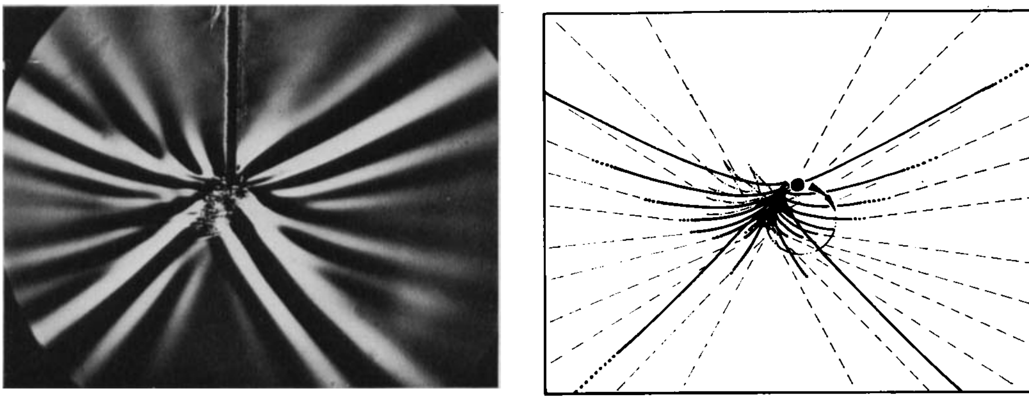


(*c*)

FIGURE 6. A horizontal cylinder moves slowly in a circular path from an impulsive start with $t_0\omega_c = 3{\cdot}14$ and $\omega_c/\omega_0 = 0{\cdot}098$. The scale lengths are 50 mm. (*a*) $t\omega_c = 6{\cdot}20$. ——, impulsive-start waves, equation (10); $\cdots$——$\cdots$, pseudo-steady waves ($\omega_f = 0$), equation (13). (*b*) $t\omega_c = 7{\cdot}22$. (*c*) $t\omega_c = 11{\cdot}30$.



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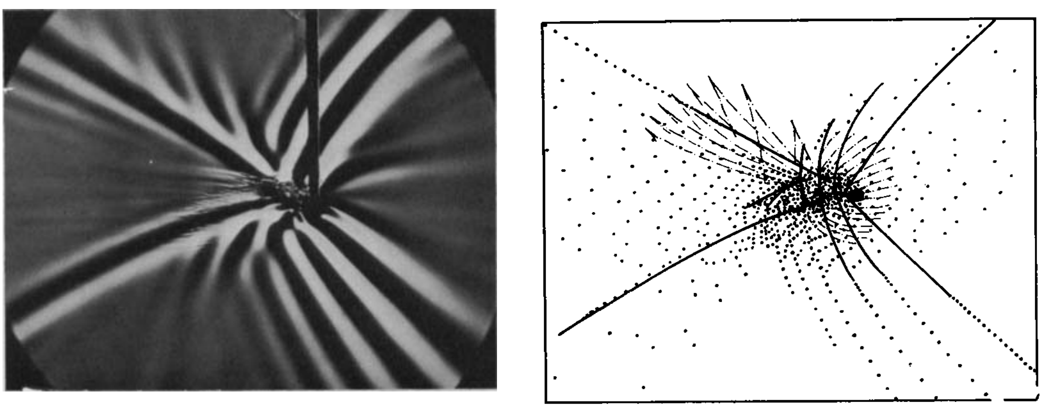

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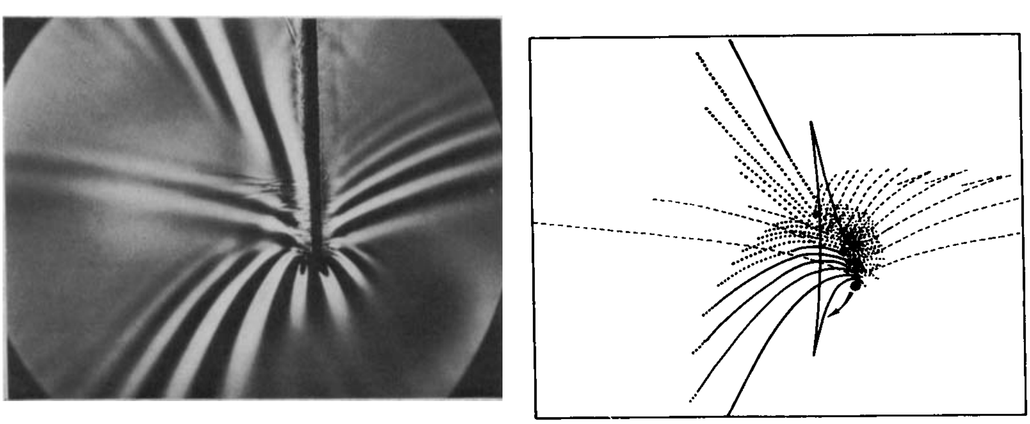

(*c*)

FIGURE 7. A horizontal cylinder in a circular path, equation (13). (*a*), (*b*) ···——···, oscillatory wave system; — - — -, first-harmonic waves; – – –, impulsive waves. $t_0\omega_c = 4\cdot71$, $\omega_c/\omega_0 = 0\cdot024$, $\omega_f/\omega_0 = 0\cdot59$. (*a*) $t\omega_c = 5\cdot67$. (*b*) $t\omega_c = 7\cdot26$. (*c*) Oscillations above the natural frequency. ···——···, oscillatory system; – – –, pseudo-steady wave system ($\omega_f = 0$). $t_0\omega_c = 0$, $t\omega_c = 1\cdot91$, $\omega_c/\omega_0 = 0\cdot024$, $\omega_f/\omega_0 = 1\cdot05$.

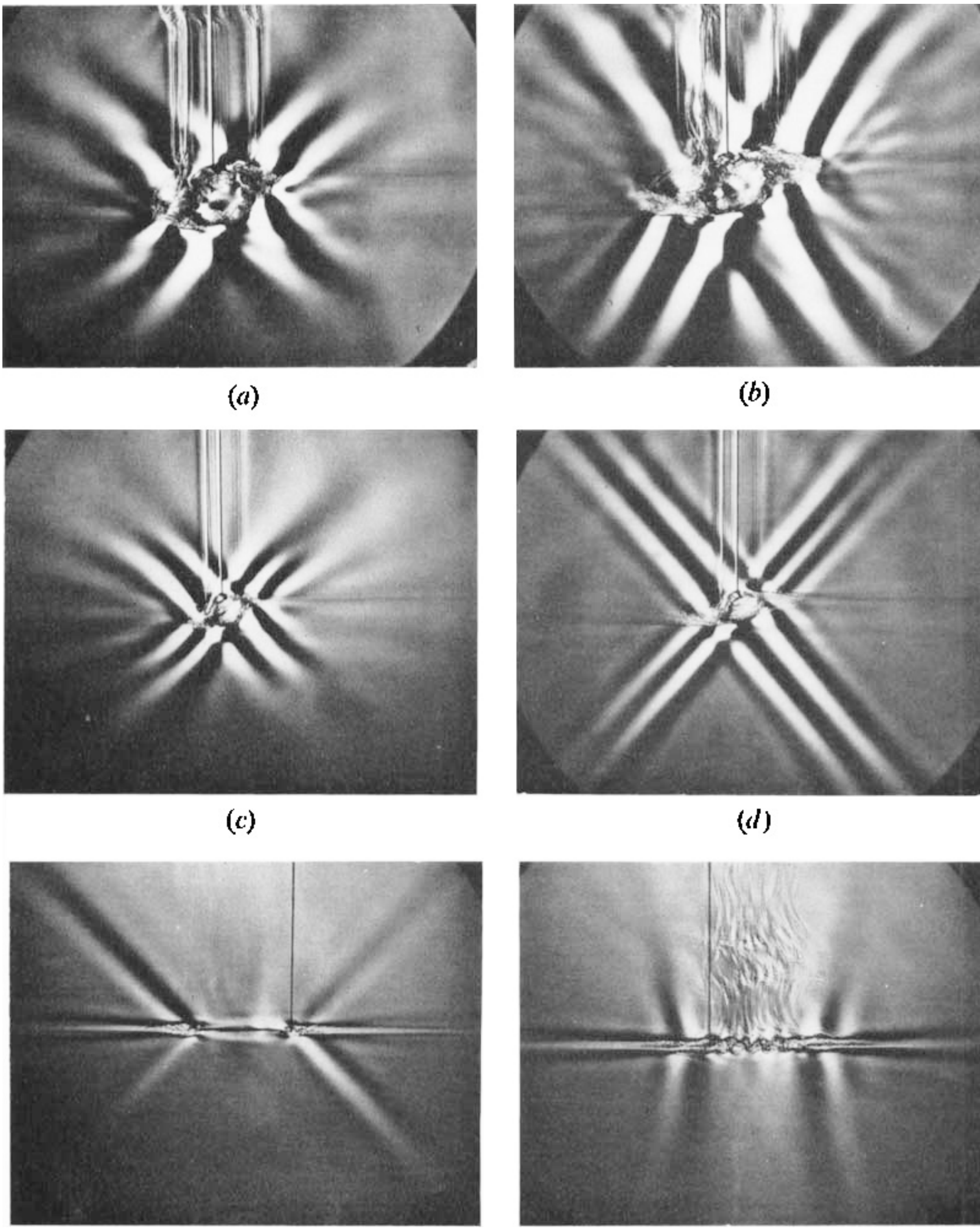

FIGURE 9. (*a*)–(*d*) A horizontal cylinder of 1 mm diameter moves in a circular path with $\omega_f = 0$, $t_0\omega_c = 0$, $\omega_0 = 1{\cdot}35\ \mathrm{rad\,s^{-1}}$. (*a*) $\omega_c/\omega_0 = 0{\cdot}89$, $R = 25{\cdot}4$ mm, $t\omega_c = 18{\cdot}9$. (*b*) $\omega_c/\omega_0 = 0{\cdot}89$, $R = 25{\cdot}4$ mm, $t\omega_c = 37{\cdot}7$. (*c*) $\omega_c/\omega_0 = 0{\cdot}75$, $R = 12{\cdot}7$ mm, $t\omega_c = 18{\cdot}9$. (*d*) $\omega_c/\omega_0 = 0{\cdot}75$, $R = 12{\cdot}7$ mm, $t\omega_c = 56{\cdot}5$. (*e*), (*f*) A horizontal cylinder of diameter 1 mm oscillates with a large amplitude in a horizontal plane. $R = 42$ mm, $\omega_0 = 1{\cdot}10\ \mathrm{rad\,s^{-1}}$, $t_0\omega_c = 0$. (*e*) $\omega_c/\omega_0 = 0{\cdot}79$, $t\omega_c = 45{\cdot}6$. (*f*) $\omega_c/\omega_0 = 0{\cdot}98$, $t\omega_c = 64{\cdot}4$.

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